EM Tutorübung – 3.5.2010 (Blatt 1)

Einführung:

vektorwertige Funktion

$\vec{f}$

$$f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$$

$$x \in \mathbb{R}^n \qquad \vec{f}(\vec{x}) = \begin{pmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{pmatrix}$$

→ Skalarfeld $\quad f : D \subseteq \mathbb{R}^n \to \mathbb{R}$

$$f(x, y, z) = x^3 + xz + \exp(y)$$

→ Vektorfeld $\quad f : D \subset \mathbb{R}^n \to \mathbb{R}^n$

$$v(x, y, z) = \begin{pmatrix} x^3 + y^2 \\ z^2 \\ 4x \end{pmatrix}$$

→ Kurve $\quad f : D \subseteq \mathbb{R} \to \mathbb{R}^n$

$$w(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \qquad t \in [0; 2\pi]$$

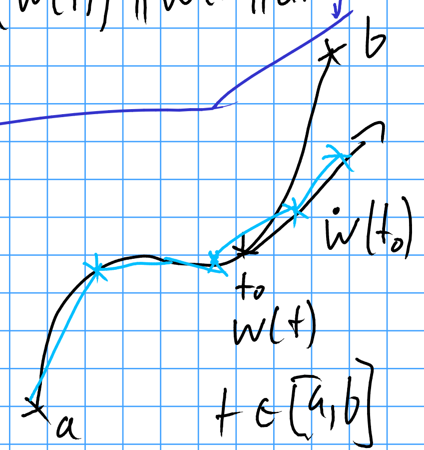
Mehrdimensionale Integralrechnung:

$\|\frac{d}{dt} w(t)\|$

Kurvenintegral 1. Art

$f: \mathbb{R}^n \to \mathbb{R}, \quad w: [a,b] \to \mathbb{R}^n$

$$\boxed{\int_w f \, ds := \int_a^b f(w(t)) \, \|\dot{w}(t)\| \, dt}$$



$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\quad y = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

3×1 $\quad 3 \times 1$ $\langle x, y \rangle = x^T y$

$= 2 + 5 + 12 = 19$

Kurvenintegral 2. Art

$v: \mathbb{R}^n \to \mathbb{R}^n, \quad w: [a,b] \to \mathbb{R}^n$

$$\boxed{\int_w v^T dx := \int_a^b \underbrace{v(w(t))^T}_{\in \mathbb{R}^n} \underbrace{\dot{w}(t)}_{\in \mathbb{R}^n} dt}$$

Aufgabe 1

$$c(t) = \begin{pmatrix} r\cos(2\pi t) \\ r\sin(2\pi t) \\ \frac{1}{3} h t \end{pmatrix} \qquad \dot{c}(t) = \begin{pmatrix} -2\pi r \sin(2\pi t) \\ 2\pi r \cos(2\pi t) \\ \frac{1}{3} h \end{pmatrix}$$

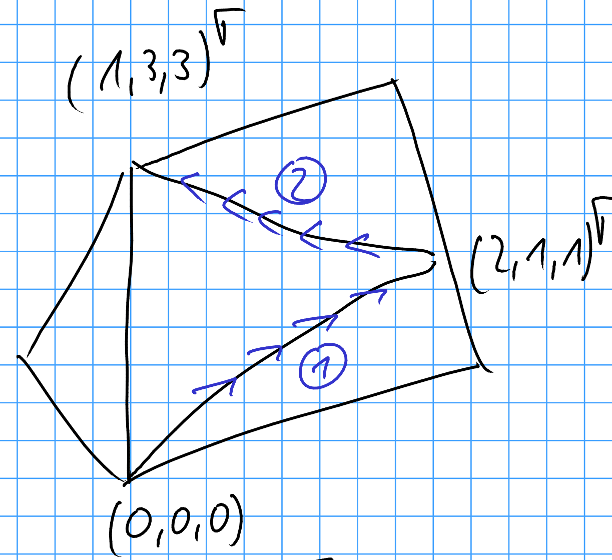
$t \in [0,3]$

Parametrisierung

$$\|\dot{c}(t)\| = \sqrt{4\pi^2 r^2 \sin^2(2\pi t) + 4\pi^2 r^2 \cos^2(2\pi t) + \frac{1}{9} h^2} = \sqrt{4\pi^2 r^2 + \frac{1}{9} h^2}$$

$$s(t) = \int_0^{t_1} \|\dot{c}(t)\|\, dt = \sqrt{4\pi^2 r^2 + \frac{1}{9} h^2} \cdot t_1 \approx 151{,}3\,\text{m}$$

Aufgabe 2



$$\vec{F}_g = -m \cdot g \cdot \vec{e}_2 = \\ = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix}$$

$$w_1(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \left[\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right] \cdot t = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \cdot t \quad t \in [0;1]$$

auch richtig: $w_1(t) = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$ $t \in [0;0{,}5]$

$$w_2(t) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \left[\begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right] \cdot t = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \quad t \in [0;1]$$

$$\dot{w}_1(t) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \qquad \dot{w}_2(t) = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

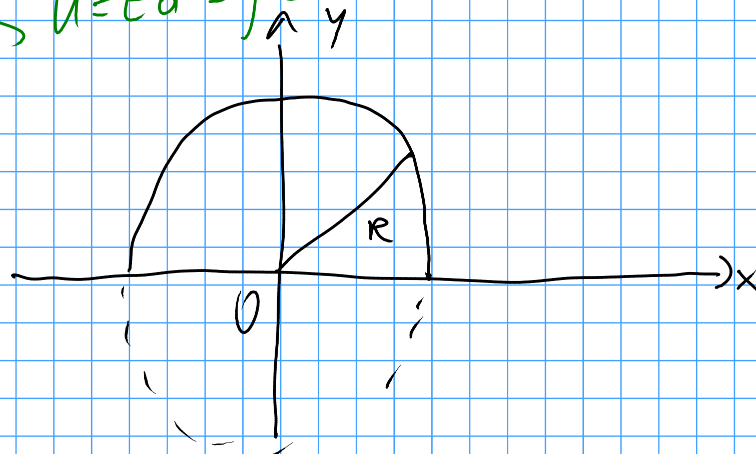
$$W = \int_{w_1} \vec{F}_g^T dx + \int_{w_2} \vec{F}_g^T dx = \int_0^1 \vec{F}_g(w_1(t))^T \dot{w}_1(t)\, dt + \int_0^1 \vec{F}_g(w_2(t))^T \cdot \dot{w}_2(t)\, dt$$

$$= \int_0^1 \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix}^T \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} dt + \int_0^1 \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} dt =$$

$$= \int_0^1 -mg \, dt + \int_0^1 -2mg \, dt = \underline{\underline{-3mg}}$$

$$E = \frac{U}{d} \Leftrightarrow U = Ed = \int \vec{E} \, d\vec{r}$$

Aufgabe 3



geg: $\vec{E}_1 = \begin{pmatrix} E_0 \\ 0 \end{pmatrix}$

$\vec{E}_2 = \begin{pmatrix} \frac{x}{x_0} E_0 \\ 0 \end{pmatrix}$

$$w(t) = \begin{pmatrix} R\cos t \\ R\sin t \end{pmatrix} \quad t \in [0; \pi] \qquad \dot{w}(t) = \begin{pmatrix} -R\sin t \\ R\cos t \end{pmatrix}$$

(a1) $$U = \int \vec{E} \, d\vec{r} = \int_0^\pi \vec{E}(w(t)) \, \dot{w}(t) \, dt = \int_0^\pi \begin{pmatrix} E_0 \\ 0 \end{pmatrix} \begin{pmatrix} -R\sin t \\ R\cos t \end{pmatrix}$$

$$= -E_0 R \int_0^\pi \sin t \, dt = -E_0 R \left[-\cos t\right]_0^\pi = E_0 R [-1-1]$$
$$= -2E_0 R$$

(a2) $$U = \ldots \int_0^\pi \begin{pmatrix} \frac{R\cos t}{x_0} E_0 \\ 0 \end{pmatrix} \begin{pmatrix} -R\sin t \\ R\cos t \end{pmatrix} dt =$$

$$= -\frac{E_0 R^2}{x_0} \int_0^\pi \cos t \sin t \, dt = -\frac{E_0 R^2}{x_0} \frac{1}{2} \left[\sin^2 t\right]_0^\pi = \underline{\underline{0}}$$