EM Tutorübung, Blatt 2 (10.5.2010)

WdH:

	Skalarfeld	$f: D \subseteq \mathbb{R}^n \to \mathbb{R}$
	Vektorfeld	$f: D \subseteq \mathbb{R}^n \to \mathbb{R}^n$
	Kurve	$f: D \subseteq \mathbb{R} \to \mathbb{R}^n$

Kurvenintegral:

1. Art: $$\int_w f \, ds := \int_a^b f(w(t)) \, \|\dot{w}(t)\| \, dt$$

f: Skalarfeld

$w: [a,b] \subseteq \mathbb{R} \to \mathbb{R}^n$

$\langle v, w \rangle := v^T w$

2. Art: $$\int_w v^T dx := \int_a^b v(w(t))^T \dot{w}(t) \, dt$$

v: Vektorfeld

HEUTE: Differenzierbarkeit

partielle Ableitung:
$f: \mathbb{R}^n \to \mathbb{R}^n$

$$\frac{\partial f}{\partial x_j} = \begin{pmatrix} \partial_{x_j} f_1 \\ \partial_{x_j} f_2 \\ \vdots \\ \partial_{x_j} f_n \end{pmatrix}$$

$$f(x,y,z) = \begin{pmatrix} 3x^2 e^y \\ 4z\sin x \\ e^x y \end{pmatrix} \qquad \frac{\partial f}{\partial y} = \begin{pmatrix} 3x^2 e^y \\ 0 \\ e^x \end{pmatrix}$$

Differentialoperatoren: (Nabla-Kalkül)

Gradient:

$f: \mathbb{R}^n \to \mathbb{R}$

$$\nabla f = \begin{pmatrix} \partial_{x_1} f \\ \partial_{x_2} f \\ \vdots \\ \partial_{x_n} f \end{pmatrix}$$

Bsp: $f(x,y,z) = x^3 e^z + \sin(z)x + 9y^2$

$$\nabla f(x,y,z) = \begin{pmatrix} 3x^2 e^z + \sin(z) \\ 18y \\ x^3 e^z + \cos(z)x \end{pmatrix} = \begin{pmatrix} \partial_x f(x,y,z) \\ \partial_y f(x,y,z) \\ \partial_z f(x,y,z) \end{pmatrix}$$

$\Rightarrow$ vgl. Plot in der mathem. Einführung (S. 2)

$f: \mathbb{R}^n \to \mathbb{R}^n$

$$\operatorname{div}(f) = \nabla^T \cdot f = \nabla \cdot f = \begin{pmatrix} \partial x_1 \\ \partial x_2 \\ \vdots \\ \partial x_n \end{pmatrix}^T \cdot \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} =$$

$$= \partial_{x_1} f_1 + \partial_{x_2} f_2 + \dots + \partial_{x_n} f_n$$

$\operatorname{div} \vec{D} = \rho$

$\operatorname{div} \vec{B} = 0$

$\operatorname{rot} \vec{H} = \vec{j} + \frac{d\vec{D}}{dt}$

$\operatorname{rot} \vec{E} = -\frac{d\vec{B}}{dt}$

Bsp: $f(x,y,z) = \begin{pmatrix} 3x^2 e^y \\ 4z\sin(x) \\ e^x y \end{pmatrix}$

$\Rightarrow$ $\boxed{\text{Quellendichte}}$

$$\operatorname{div} f(x,y,z) = 6xe^y + 0 + 0 = \underline{\underline{6xe^y}}$$

$f: \mathbb{R}^3 \to \mathbb{R}^3$

$$\operatorname{rot} f = \nabla \times f = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \times \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} \partial_2 f_3 - \partial_3 f_2 \\ \partial_3 f_1 - \partial_1 f_3 \\ \partial_1 f_2 - \partial_2 f_1 \end{pmatrix}$$

$\Rightarrow$ **Wirbelhaltigkeit**

Mehrfachintegrale:

$$\int_{z=c}^{d} \cdots \int_{v=} \int_{u=a}^{b} f(u, \dots, z)\, du\, dv \dots dz$$

Auswertung von innen nach außen!

Aufgabe 4

$$I = \int_0^{x_0} \int_0^{y_0} \int_0^{z_0} x^2 \sin y \, dz\, dy\, dx =$$

$$= \int_0^{x_0} \int_0^{y_0} \left[x^2 \sin y \, z \right]_0^{z_0} dy\, dx = z_0 \int_0^{x_0} \int_0^{y_0} (x^2 \sin y)\, dy\, dx =$$

$$= z_0 \int_0^{x_0} \left[-x^2 \cos y\right]_0^{y_0} dx = -z_0 \int_0^{x_0} x^2 (\cos y_0 - 1)\, dx =$$

$$= +z_0 \int_0^{x_0} x^2 - x^2 \cos y_0 \, dx = z_0 \left[\frac{1}{3}x^3 - \frac{1}{3}x^3 \cos y_0\right]_0^{x_0}$$

$$= \frac{1}{3} z_0 x_0^3 (1 - \cos y_0)\Big|_{\substack{y_0 = \pi \\ z_0 = 2\pi}} = \frac{1}{3} \cdot 2\pi x_0^3 (1+1)$$

$$= \frac{4}{3} \pi x_0^3$$

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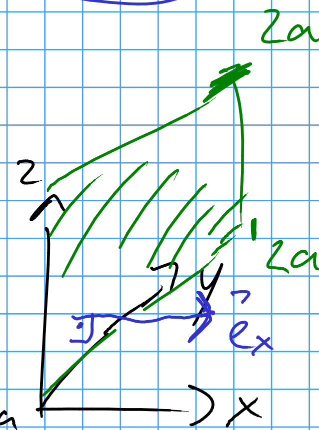
Volumenelement in Kugelkoordinaten

$$dV = r^2 \sin\vartheta \, dr \, d\varphi \, d\vartheta$$

$$V_{Kugel} = \iiint dV$$

$da = dy\,dz$

Aufgabe 5 ges: $\vec{F} = \iint_A \vec{f}_\perp \, da$



a) $\vec{f}_a = f_0 \cdot \vec{e}_x$

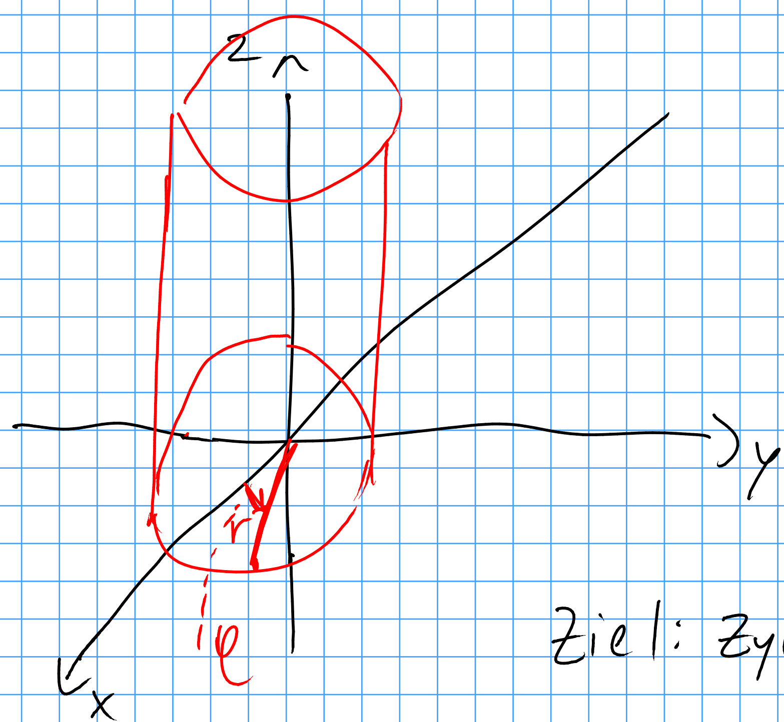
$$\vec{F} = \int_0^{2a} \int_0^{2a} f_0 \cdot \vec{e}_x \, dy\,dz = f_0 \vec{e}_x \int_0^{2a}\int_0^{2a} dy\,dz = f_0 \, 4a^2 \, \vec{e}_x$$

b) $\vec{f}_a = \frac{f_0}{\sqrt{2}}(\vec{e}_x + \vec{e}_y) \Rightarrow \vec{f}_\perp = \frac{f_0}{\sqrt{2}}\vec{e}_x$

$$\vec{F} = \int_0^{2a}\int_0^{2a} \frac{f_0}{\sqrt{2}}\vec{e}_x\, dy\, dz = \frac{f_0}{\sqrt{2}}\vec{e}_x \int_0^{2a}\int_0^{2a} dy\, dz = \underline{\underline{\frac{f_0}{\sqrt{2}} \cdot 4a^2 \vec{e}_x}}$$

6. Aufgabe

$$\vec{x}(t) = R\cos(2\pi t)\vec{e}_x + R\sin(2\pi t)\vec{e}_y + \frac{1}{3}ht\vec{e}_z \quad (*)$$



Zylinderkoordinaten

||

Polarkoordinaten (r, φ)

\+

dritte Komponente

(z-Komponente)

Ziel: Zylinderkoordinaten

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$$\vec{e}_r(\varphi) = \vec{e}_x \cos\varphi + \vec{e}_y \sin\varphi$$

$$(*)\quad \vec{x}(t) = R\vec{e}_r(2\pi t) + \frac{1}{3}ht\vec{e}_z$$

Aufgabe 7

a) $f(x_1, x_2) = x_1^2 + 3x_1x_2$

$$\nabla f = \begin{pmatrix} 2x_1 + 3x_2 \\ 3x_1 \end{pmatrix}$$

b) $g(x,y,z) = \sin(x^2) + 2e^y$

$$\nabla g = \begin{pmatrix} \cos(x^2) \cdot 2x \\ 2e^y \\ e^y \end{pmatrix}$$

c) $h(x,y,z) = e^x yz$

$$\nabla h = \begin{pmatrix} e^x yz \\ e^x z \\ e^x y \end{pmatrix}$$