EM-Tutorübung, 31.05.2010

(Springer S. 249

$u: \mathbb{R}^n \to \mathbb{R} \qquad \nabla u = \frac{\partial u}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial u}{\partial \vartheta} e_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial u}{\partial \varphi}\vec{e}_\varphi$)

WdH: Coulomb-Gesetz: $\vec{F}_{el} = \frac{q}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{q_i}{\|\vec{r}-\vec{r}_i\|^3}(\vec{r}-\vec{r}_i)$

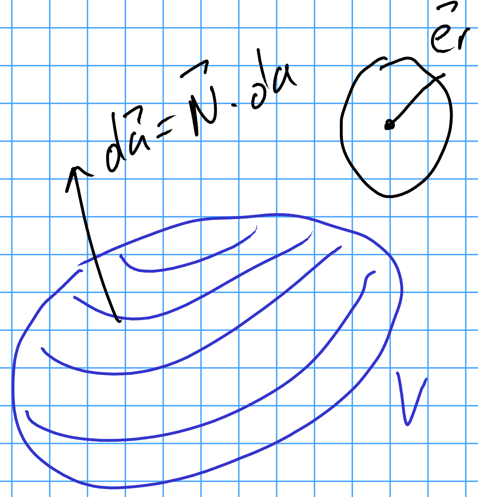
E-Feld: $\vec{E} = \frac{\vec{F}_{el}}{q} = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{q_i}{\|\vec{r}-\vec{r}_i\|^3}(\vec{r}-\vec{r}_i)$

Potential: $\phi(\vec{r}) = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{q_i}{\|\vec{r}-\vec{r}_i\|}$

Raumladung: $[\rho] = \frac{C}{m^3}$ (Analogie $\rho = \frac{m}{V}$)

Somit $Q(V) = \iiint_V \rho \, dV$

$\Rightarrow$ Verschiebungsfluss: $\vec{D} = \underbrace{\varepsilon_0\varepsilon_r}_{\varepsilon}\vec{E} = \varepsilon\vec{E} = \Big|_{\text{Spezialfall}}\ \varepsilon \cdot \frac{1}{4\pi\varepsilon}\frac{q}{r^2}\cdot\vec{e}_r$



$\Rightarrow$ materialunabhängig

$= \frac{1}{4\pi}\frac{q}{r^2}\vec{e}_r$

$\iint_{\partial V} \vec{D}\, d\vec{a} = Q(V)$

$\partial V \;\hat{=}$ Rand des Volumens $V \qquad \partial V = M$

Beispiel:

$$\iint_{\partial V} \vec{D}\, d\vec{a} = \int_0^{2\pi}\int_0^{\pi} \frac{1}{4\pi}\frac{q}{r^2}\,\underbrace{\vec{e}_r\,\vec{e}_r}_{1}\, r^2 \sin\vartheta\, d\vartheta\, d\varphi =$$

$$= \frac{q}{4\pi}\int_0^{2\pi}\int_0^{\pi} \sin\vartheta\, d\vartheta\, d\varphi = \frac{q}{4\pi}\cdot 4\pi = \underline{\underline{q}}$$

$$Q(V) = Q(V)$$

$$\iiint_V \rho(\vec{r})\, dV = \iint_{\partial V} \vec{D}\, d\vec{a}$$

Satz v. Gauß

$$\iiint_V \rho(\vec{r})\, dV = \iiint_V \operatorname{div} \vec{D}\, dV$$

$$\operatorname{div} \vec{D} = \rho$$ Gaußsches Gesetz

→ folgt aus dem Umstand, dass zwei Integrale nur dann gleich sind, wenn auch ihre Integranden gleich sind

Aufgabe 12

d)
$$\iint_{\partial V} \vec{E}\, d\vec{a} = \int_0^{2\pi}\int_0^{\pi} \frac{q}{4\pi\varepsilon_0\varepsilon_r}\cdot\frac{1}{r^2}\cdot\vec{e}_r\,\underbrace{\vec{e}_r\, r^2 \sin\vartheta\, d\vartheta\, d\varphi}_{d\vec{a}}$$

$$= \frac{q}{4\pi\varepsilon_0\varepsilon_r}\cdot\int_0^{2\pi}\int_0^{\pi} \sin\vartheta\, d\vartheta\, d\varphi = \underline{\underline{\frac{q}{\varepsilon_0\varepsilon_r}}}$$

vgl. $\iint_{\partial V} \vec{D}\, d\vec{a} = q$; mit $\vec{E} = \frac{1}{\varepsilon_0\varepsilon_r}\vec{D}$

c) $\boxed{\iint \vec{D}\, d\vec{a} = q}$ (Rechnung vollkommen Analog für

$$\vec{D} = \frac{1}{4\pi}\frac{q}{r^2}\vec{e}_r$$

f) ges: ρ, sodass $q = \iiint_V \rho(\vec{r})\, dV$

tatsächlich: $\rho(\vec{r}) = \rho = \text{const.}$

$$q = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} \rho\, r^2 \sin\vartheta\, dr\, d\varphi\, d\vartheta = \rho \underbrace{\int_0^{\pi}\int_0^{2\pi}\int_0^{1} r^2 \sin\vartheta\, dr\, d\varphi\, d\vartheta}_{\text{Volumen einer Kugel}} =$$

$$= \rho\,\frac{4}{3}\pi \quad \Rightarrow \quad \rho = \frac{3}{4\pi}q$$

Hinweise zur Aufgabe 13:

a) $Q = \iiint \rho(\vec{r})\, dV$

b) $\vec{D}(\vec{r}) = D_r(r)\cdot\vec{e}_r$

$Q(V) = \iint_{\partial V} \vec{D}(\vec{r})\, d\vec{a}$

d) $\phi(\vec{r}) = -\int_{-\infty}^{r} \vec{E}\, d\vec{r} + \underbrace{\phi(r=\infty)}_{=0}$

13. Aufgabe

a) $$\rho(\vec{r}) = \begin{cases} \rho_0 \left(1 - \frac{r^2}{a^2}\right) & r \leq a \\ 0 \end{cases}$$

ges: Q Fall 1: $r < a$

Fall 2: $r \geq a$

Lös: $$Q_1(V) = \iiint \rho_1(\vec{r})\, dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{r} \rho_0 \left(1 - \frac{r'^2}{a^2}\right) r'^2 \sin\vartheta \, dr' \, d\vartheta \, d\varphi$$

$$= \rho_0 \, 4\pi \left(\frac{1}{3} r^3 - \frac{1}{5} \frac{r^5}{a^2}\right)$$

$$Q_2(V) = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho_1(\vec{r})\, dV + \underbrace{\int_0^{\pi}\int_0^{2\pi}\int_a^{r} \underbrace{\rho_2(\vec{r})}_{=0} dV}_{=0}$$

$$= Q_1(V)\Big|_{r=a} = \rho_0 \, 4\pi \, \frac{2}{15} a^3$$

b) ges: $D_r(r)$

Lös: $$Q(V) = \iint_{\partial V} D(\vec{r})\, d\vec{a}$$

$$Q_1(r) = \iint_{\partial V} \vec{D}_1(\vec{r})\, d\vec{a} = \int_0^{2\pi}\int_0^{\pi} D_{r_1}(r) \cdot \vec{e}_r \cdot \vec{e}_r r^2 \sin\vartheta \, d\vartheta \, d\varphi$$

$$= D_{r_1}(r)\, r^2 \int_0^{2\pi}\int_0^{\pi} \sin\vartheta \, d\vartheta\, d\varphi = 4\pi\, D_{r_1}(r)\, r^2$$

$$\Rightarrow D_{r_1}(r) = \frac{Q_1(r)}{4\pi r^2} = \frac{\varrho_0 4\pi \left(\frac{1}{3}r^3 - \frac{1}{5}\frac{r^5}{a^2}\right)}{4\pi r^2} =$$

$$= \varrho_0 \left(\frac{1}{3} r - \frac{1}{5}\frac{r^3}{a^2}\right)$$

<u>Fall 2:</u> $$Q_2(r) = \iint_{\partial V} \vec{D}_2(r)\, d\vec{a} = \int_0^{2\pi}\int_0^{\pi} D_2(r)\, r^2 \sin\vartheta\, d\vartheta\, d\varphi$$

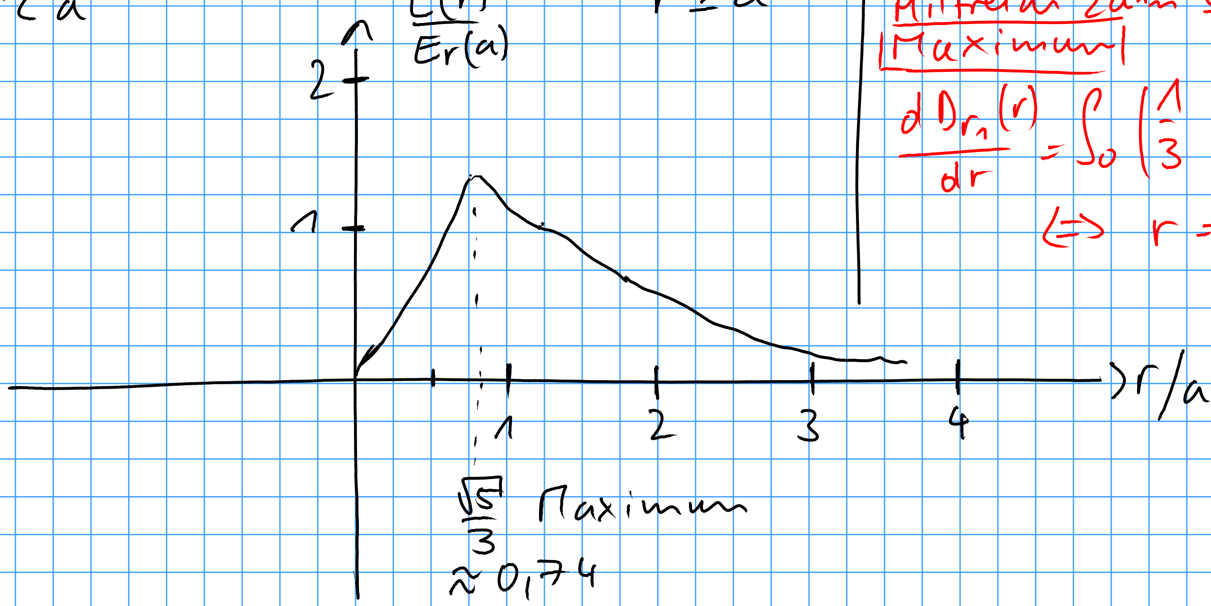
$$= D_2(r) \cdot 4\pi r^2$$

$$\Rightarrow D_2(r) = \frac{Q_2(r)}{4\pi r^2} = \frac{\varrho_0\, 4\pi \frac{2}{15} a^3}{4\pi\, r^2} = \varrho_0 \frac{2}{15}\frac{a^3}{r^2}$$

c) $$E_r(r) = \frac{D_r(r)}{\varepsilon}$$

$$D_{r_1}(r) = \varrho_0 \left(\frac{r}{3} - \frac{r^3}{5a^2}\right) \qquad D_{r_2} = \frac{2}{15}\varrho_0 \frac{a^3}{r^2}$$

$r < a$ $\qquad$ $r \geq a$



<u>Hilfreich zum Skizzieren: Maximum</u>

$$\frac{d D_{r_1}(r)}{dr} = \varrho_0 \left(\frac{1}{3} - \frac{3}{5}\frac{r^2}{a^2}\right) \stackrel{!}{=} 0$$

$$\Leftrightarrow r = (\pm)\, a \frac{\sqrt{5}}{3}$$

d) $\phi(r) = -\int_{\infty}^{r} \vec{E}_2 d\vec{r} + \underbrace{\phi(r=\infty)}_{=0} = -\int_{\infty}^{r} \vec{E}\, dr =$ $\quad \vec{D} = \varepsilon \vec{E} \Rightarrow \vec{E} = \frac{\vec{D}}{\varepsilon}$

$$= -\int_{\infty}^{r} \frac{2}{15} \frac{\rho_0}{\varepsilon} \frac{a^3}{r'^2} \vec{e}_r \cdot \vec{e}_r \, dr' = \frac{2}{15} \frac{\rho_0}{\varepsilon} \frac{a^3}{r}$$

Fall 1 $a \leq r \leq \infty$

Fall 2: $0 \leq r \leq a$

$$\phi(r) = -\int_{\infty}^{a} \vec{E}_2 d\vec{r} - \int_{a}^{r} \vec{E}_1 \, d\vec{r} =$$

$$= \frac{2}{15} \frac{\rho_0}{\varepsilon} a^2 - \int_{a}^{r} \frac{\rho_0}{\varepsilon} \left(\frac{r'}{3} - \frac{r'^3}{5a^2} \right) \vec{e}_r \cdot \vec{e}_r \, dr' =$$

$$= \frac{2}{15} \frac{\rho_0}{\varepsilon} a^2 - \frac{\rho_0}{\varepsilon} \left[\frac{1}{2} \frac{r'^2}{3} - \frac{r'^4}{20a^2} \right]_a^r =$$

$$= \frac{2}{15} \frac{\rho_0}{\varepsilon} a^2 - \frac{\rho_0}{\varepsilon} \left[\frac{1}{6} r^2 - \frac{r^4}{20a^2} - \frac{1}{6} a^2 + \frac{a^2}{20} \right] =$$

$$= \frac{2}{15} \frac{\rho_0}{\varepsilon} a^2 + \frac{\rho_0 7a^2}{\varepsilon 60} - \frac{\rho_0}{\varepsilon} \left[\frac{r^2}{6} - \frac{r^4}{20a^2} \right] =$$

$$= \frac{1}{4} \frac{\rho_0}{\varepsilon} a^2 - \frac{\rho_0}{\varepsilon} \left[\frac{r^2}{6} - \frac{r^4}{20a^2} \right]$$

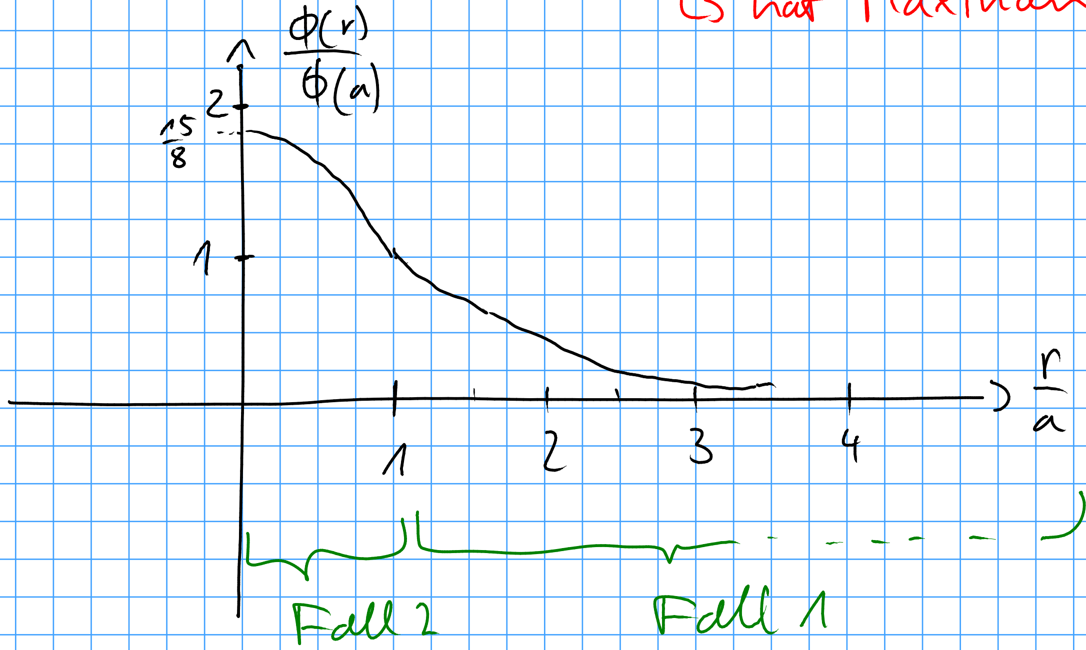
$$\phi(a) = \frac{2}{15} \frac{\rho_0}{\varepsilon} a^2$$

Fall 1: $r > a$ $\quad \frac{\phi(r)}{\phi(a)} = \frac{2/15 \frac{\rho_0}{\varepsilon} \frac{a^3}{r}}{2/15 \frac{\rho_0}{\varepsilon} a^2} = \frac{a}{r}$

Fall 2: $0 < r < a$ $\quad \frac{\phi(r)}{\phi(a)} = \frac{\frac{1}{4} \frac{\rho_0}{\varepsilon} a^2 - \frac{\rho_0}{\varepsilon} \left[\frac{r^2}{6} - \frac{r^4}{20a^2} \right]}{2/15 \frac{\rho_0}{\varepsilon} a^2}$

$$= \frac{15}{8} - \frac{15}{2}\cdot\frac{1}{6}\left(\frac{r}{a}\right)^2 + \frac{15}{2}\cdot\frac{1}{20}\left(\frac{r}{a}\right)^4$$

↳ hat Maximum bei 0



f) ges: Spannung U_a (Potentialdifferenz)

Lös: $U_{0a} = \phi(0) - \phi(a) = \frac{1}{4}\frac{\rho_0}{\varepsilon}a^2 - \frac{2}{15}\frac{\rho_0}{\varepsilon}a^2 = \underline{\underline{\frac{7}{60}\frac{\rho_0}{\varepsilon_0}a^2}}$