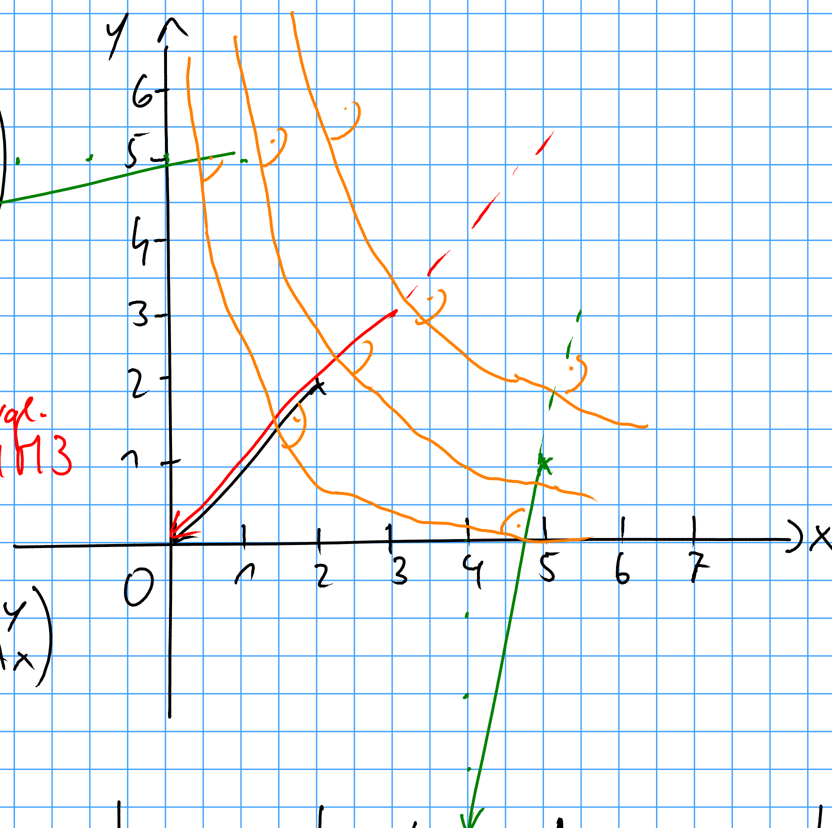
EM-Tutorübung, Blatt VII, 14.06.2010

Einführung/WdH:

$\vec{E} = \begin{pmatrix} -Ay \\ -Ax \end{pmatrix} := \begin{pmatrix} -y \\ -x \end{pmatrix}$

A=1



$\vec{E} = -\nabla \Phi$

Stammfkt, vgl. HM3

Beh: $\phi = Axy$

$\nabla \Phi = \begin{pmatrix} \partial_x \phi \\ \partial_y \phi \end{pmatrix} = \begin{pmatrix} Ay \\ Ax \end{pmatrix}$

Äquipotentiallinie: $\phi = \text{const.} \Leftrightarrow Axy = \text{const.} = C$

$y = \frac{C}{Ax}$

Aufgabe 16:

$$\vec{E} = \frac{1}{x^2+y^2+z^2} \left(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z \right)$$

$$\text{div}\,\vec{E} = \nabla \cdot \vec{E} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \cdot \vec{E} = \partial_x E_x + \partial_y E_y + \partial_z E_z =$$

$$= \frac{x^2+y^2+z^2-2x^2}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2+z^2-2y^2}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2+z^2-2z^2}{(x^2+y^2+z^2)^2} =$$

$$= \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \frac{1}{x^2+y^2+z^2}$$

Zylinderkoordinaten: $r = \sqrt{x^2+y^2}$

$$\operatorname{div} \vec{E} = \frac{1}{r^2+z^2}$$

Kugelkoordinaten: $r = \sqrt{x^2+y^2+z^2}$

$$\operatorname{div} \vec{E} = \frac{1}{r^2}$$

Alternativ (Zylinderkoordinaten)

$$\vec{E} = \frac{1}{x^2+y^2+z^2} \left(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z \right) =$$

$$= \frac{1}{r^2+z^2} \Big(r\cos\varphi \left(\vec{e}_r \cos\varphi - \vec{e}_\varphi \sin\varphi \right) + r\sin\varphi \left(\vec{e}_r \sin\varphi + \vec{e}_\varphi \cos\varphi \right) + z\vec{e}_z \Big) =$$

$$= \frac{1}{r^2+z^2} \left(r\cos^2\varphi\, \vec{e}_r + r\sin^2\varphi\, \vec{e}_r + z\vec{e}_z \right) =$$

$$= \frac{1}{r^2+z^2} \left(r\vec{e}_r + z\vec{e}_z \right)$$

$$\operatorname{div} \vec{E} = \frac{1}{r}\frac{\partial (r E_r)}{\partial r} + \underbrace{\frac{1}{r}\frac{\partial E_\varphi}{\partial \varphi}}_{=0} + \frac{\partial E_z}{\partial z}$$

Aufgabe 17

$$\varepsilon(r) = \begin{cases} \varepsilon_1 \varepsilon_0 & 0 \le r < 2R \\ \varepsilon_2 \varepsilon_0 & r \ge 2R \end{cases}$$

$$\rho(r) = \begin{cases} Q \cdot r \;, & 0 < r < R \\ 0 \;, & r \ge R \end{cases}$$

(a) $\vec{E}(\vec{r}) = E_r(r)\,\vec{e}_r$

(b) $$Q = \iint_{\partial V} \vec{D}\, d\vec{a} = \iiint_V \rho(\vec{r})\, dV$$

$$\vec{D} = \varepsilon \vec{E}(r)$$

1. Fall : $0 < r < R$

$$Q_1 = \iiint \rho(r)\, dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{r} Q \cdot r' \cdot r'^2 \sin\vartheta \, dr' d\varphi \, d\vartheta =$$

$$= 4\pi Q \int_0^r r'^3 \, dr = 4\pi Q \cdot \frac{1}{4} r^4 = Q\pi r^4$$

2. Fall : $r > R$

$Q_2 = Q_1(r)\big|_{r=R} = Q\pi R^4$

Übergang zum E-Feld:

$$\iint \vec{D}\, d\vec{a} = Q_{eing.}$$

$$\int_0^{\pi}\int_0^{2\pi} D(r)\,\vec{e}_r \cdot \vec{e}_r\, r^2 \sin\vartheta \, d\varphi\, d\vartheta = Q_{eing.}$$

$$\Rightarrow D_r(r) = \frac{Q_{eing.}}{4\pi r^2}$$

1. Fall: $0 < r < R$

$$D_r(r) = \frac{Q\pi r^4}{4\pi r^2} = \frac{Q\pi r^2}{4\pi} = \frac{Qr^2}{4}$$

$$\Rightarrow E_r(r) = \frac{Qr^2}{4\varepsilon_0\varepsilon_1}$$

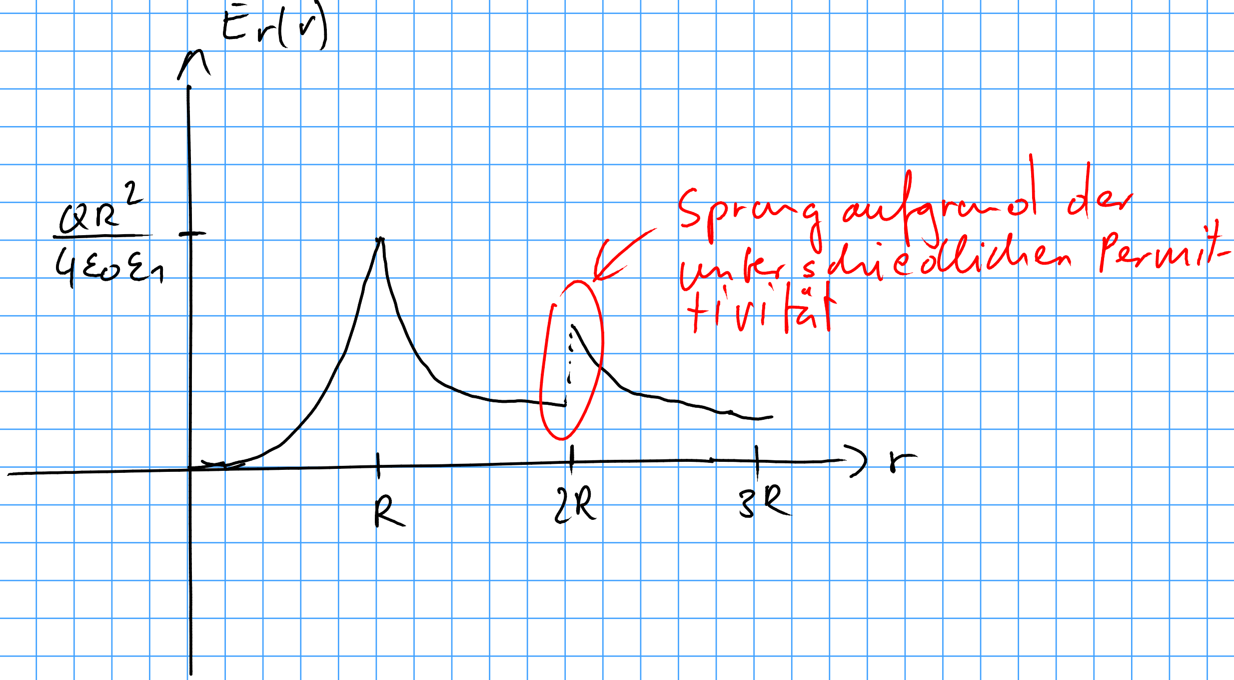
2. Fall: $r > R$

a) $$D_r(r) = \frac{Q\pi R^4}{4\pi r^2} = \frac{QR^4}{4r^2}$$

Fall: $R < r < 2R$: $E_r(r) = \frac{QR^4}{4r^2\varepsilon_0\varepsilon_1}$

Fall: $r > 2R$: $E_r(r) = \frac{QR^4}{4r^2\varepsilon_0\varepsilon_2}$

Skizze:



c) ges: Potential

Lös: 1. Fall: $r > 2R$

$$\phi_1(r) = -\int_\infty^r \frac{QR^4}{\varepsilon_0\varepsilon_2 4 r'^2}\, dr' = -\frac{QR^4}{\varepsilon_0\varepsilon_2 4}\int_\infty^r \frac{1}{r'^2}\, dr' =$$

$$= -\frac{QR^4}{\varepsilon_0\varepsilon_2 4}\left[-\frac{1}{r'}\right]_\infty^r = \frac{QR^4}{\varepsilon_0\varepsilon_2 4 r}$$

$$\phi = -\int E\, dr$$

2. Fall $2R > r > R$

$$\phi_2(r) = \phi_1(r)\Big|_{r=2R} - \int_{2R}^r \frac{QR^4}{\varepsilon_0\varepsilon_1 4 r'^2}\, dr =$$

$$= \frac{QR^3}{8\varepsilon_0\varepsilon_2} + \frac{QR^4}{\varepsilon_0\varepsilon_1 4}\left[+\frac{1}{r'}\right]_{2R}^r =$$

$$= \frac{QR^3}{8\varepsilon_0\varepsilon_2} + \frac{QR^4}{\varepsilon_0\varepsilon_1 4}\left[\frac{1}{r} - \frac{1}{2R}\right] =$$

$$= \frac{QR^3}{4\varepsilon_0}\left[\frac{1}{2\varepsilon_2} + \frac{R}{\varepsilon_1 r} - \frac{1}{2\varepsilon_1}\right]$$

d) $W_{el} = \int \vec{F}_{el}\, d\vec{r} = \qquad \vec{E} = \frac{\vec{F}_{el}}{q}$

$= 0 \quad \Rightarrow$ Bewegung auf einer Äqui-potentiallinie

18. Aufgabe

geg: $\phi(r, \vartheta, \varphi) = \frac{1}{4\pi\varepsilon}\frac{q}{r}\left(e^{-\alpha r}\left(1 + \frac{\alpha r}{2}\right) - 1\right)$

a) ges: $\vec{E}$

Lös: $\vec{E} = -\nabla\Phi$ mit $\nabla\Phi = \frac{\partial\Phi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\vartheta}\vec{e}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial\Phi}{\partial\varphi}\vec{e}_\varphi$

in Kugelkoordinaten

Es folgt aufgrund der geg. Struktur von ϕ:

$\frac{\partial\Phi}{\partial\vartheta} = 0 \quad , \quad \frac{\partial\Phi}{\partial\varphi} = 0$

$$\frac{\partial \Phi}{\partial r} = \frac{q}{4\pi\varepsilon}\left[-\frac{1}{r^2}\left(e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)-1\right) + \right.$$

$$\left. + \frac{1}{r}\left(-\alpha e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right) + \frac{\alpha}{2}e^{-\alpha r} - \frac{\alpha^2 r}{2}e^{-\alpha r}\right)\right] =$$

$$= \frac{q}{4\pi\varepsilon}\left[-\frac{1}{r^2}e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)+\frac{1}{r^2}-\frac{\alpha}{2r}e^{-\alpha r}-\frac{\alpha^2}{2}e^{-\alpha r}\right] =$$

$$= \frac{q}{4\pi\varepsilon}\left[-\frac{1}{r^2}e^{-\alpha r}-\frac{\alpha}{2r}e^{-\alpha r}+\frac{1}{r^2}-\frac{\alpha}{2r}e^{-\alpha r}-\frac{\alpha^2}{2}e^{-\alpha r}\right] =$$

$$= \frac{q}{4\pi\varepsilon}\left[e^{-\alpha r}\left(-\frac{1}{r^2}-\frac{\alpha}{r}-\frac{\alpha^2}{2}\right)+\frac{1}{r^2}\right]$$

$$\Rightarrow \vec{E} = -\nabla\Phi = \frac{q}{4\pi\varepsilon}\left[e^{-\alpha r}\left(\frac{1}{r^2}+\frac{\alpha}{r}+\frac{\alpha^2}{2}\right)-\frac{1}{r^2}\right]\vec{e}_r$$

b) mit Hilfe des Gauß'schen Gesetz:

$$Q(r) = \iint_{\partial V} \vec{D}\, d\vec{a} = \iint_{\partial V} \varepsilon \vec{E}\, d\vec{a} = \varepsilon \iint_{\partial V} \vec{E}\, d\vec{a} =$$

$$= \frac{q}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left[e^{-\alpha r}\left(\frac{1}{r^2}+\frac{\alpha}{r}+\frac{\alpha^2}{2}\right)-\frac{1}{r^2}\right]\vec{e}_r\vec{e}_r r^2 \sin\vartheta\, d\vartheta\, d\varphi =$$

$$= \frac{q}{4\pi}\cdot 4\pi r^2\left[e^{-\alpha r}\left(\frac{1}{r^2}+\frac{\alpha}{r}+\frac{\alpha^2}{2}\right)-\frac{1}{r^2}\right] =$$

$$= q\cdot\left[e^{-\alpha r}\left(1+\alpha r+\frac{\alpha^2 r^2}{2}\right)-1\right]$$

gesamte eingeschlossene Ladung:

$$Q(r)\big|_{r=R} = q\left[e^{-\alpha R}\left(1+\alpha R+\frac{\alpha^2 R^2}{2}\right)-1\right]$$

Alternative über Raumladungsdichte:

$\operatorname{div}\vec{D} = \rho$ mit $\vec{D} = \varepsilon\vec{E}$ folgt:

$$\rho = \varepsilon \operatorname{div}\vec{E} \quad \text{mit} \quad \operatorname{div}\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \underbrace{\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta E_\vartheta\right)}_{=0} + \underbrace{\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}E_\varphi}_{=0}$$

$$\rho = \varepsilon\operatorname{div}\vec{E} = \varepsilon\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) =$$

$$= \frac{\varepsilon}{r^2}\frac{\partial}{\partial r}\left(\frac{q}{4\pi\varepsilon}r^2\left[e^{-\alpha r}\left(\frac{1}{r^2}+\frac{\alpha}{r}+\frac{\alpha^2 r^2}{2}\right)-\frac{1}{r^2}\right]\right) =$$

$$= \frac{q}{4\pi r^2}\frac{\partial}{\partial r}\left[e^{-\alpha r}\left(1+\alpha r+\frac{\alpha^2 r^2}{2}\right)-1\right] =$$

$$= \frac{q}{4\pi r^2}\left[-\alpha e^{-\alpha r}\left(1+\alpha r+\frac{\alpha^2 r^2}{2}\right)+e^{-\alpha r}\left(\alpha+\alpha^2 r\right)\right] =$$

$$= \frac{q}{4\pi r^2}\left[-\alpha e^{-\alpha r} - \alpha^2 r e^{-\alpha r} - \frac{\alpha^3 r^2}{2}e^{-\alpha r} + \alpha e^{-\alpha r} + \alpha^2 r e^{-\alpha r}\right]$$

$$= -\frac{q}{4\pi r^2}\frac{\alpha^3 r^2}{2}e^{-\alpha r} = -\frac{q}{4\pi}\frac{\alpha^3}{2}e^{-\alpha r}$$

$$\Rightarrow Q(R) = \iiint \rho(r)\, dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} -\frac{q}{4\pi}\frac{\alpha^3}{2} e^{-\alpha r} r^2 \sin\vartheta \, dr\, d\varphi\, d\vartheta =$$

$$= -\frac{q}{4\pi}\frac{\alpha^3}{2}\cdot 4\pi \int_0^R r^2 e^{-\alpha r}\, dr =$$

Stammfkt von $\int r^2 e^{-\alpha r}$ laut FS (oder mittels part. Integration)

$$\int r^2 e^{-\alpha r}\, dr = -e^{-\alpha r}\,\frac{\alpha^2 r^2 + 2\alpha r + 2}{\alpha^3}$$

$$= -\frac{q}{4\pi}\frac{\alpha^3}{2}\cdot 4\pi \left[-\frac{e^{-\alpha r}\left(\alpha^2 r^2 + 2\alpha r + 2\right)}{\alpha^3}\right]_0^R =$$

$$= q\left[e^{-\alpha r}\left(1 + \alpha R + \frac{R^2\alpha^2}{2}\right) - 1\right]$$

c) Superpositionsprinzip: (mit $q = Q$)

$$\phi^* = \phi_{alt} + \phi_{neu} \quad \text{mit} \quad \phi_{neu} = \frac{1}{4\pi\varepsilon}\frac{Q}{r}$$

(Potential einer Punktladung im Ursprung)

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\left(e^{-\alpha r}\left(1 + \frac{\alpha r}{2}\right)\right)$$

$$E_r^* = E_r + \frac{1}{4\pi\varepsilon}\frac{Q}{r^2} = \frac{Q}{4\pi\varepsilon}\left[e^{-\alpha r}\left(\frac{1}{r^2} + \frac{\alpha}{r} + \frac{\alpha^2}{2}\right)\right]$$