EMF-Tutorübung, Blatt 3, 19.11.2010

Wiederholung:

$$\Delta \Phi = -\rho/\varepsilon \iff \Phi = \frac{1}{4\pi\varepsilon} \iiint_V \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \underbrace{d^3r'}_{dV}$$

$$\left(\Delta - \cancel{\varepsilon\mu \frac{\partial^2}{\partial t^2}}\right)\vec{A} = -\mu\vec{j}$$

(→ Maxwellsche Gl. mit Vektorpotentialen
→ Lorenz-Eichung)

stationär: $\frac{\partial}{\partial t} = 0$

$$\Rightarrow \Delta\vec{A} = -\mu\vec{j} \Rightarrow \vec{A} = \frac{\mu}{4\pi} \iiint_V \frac{\vec{j}(\vec{r})}{|\vec{r}-\vec{r}'|} \underbrace{d^3r'}_{dV}$$

kartesische Koordinaten:

$$\Delta\vec{A} = -\mu\vec{j}$$

$$\iff \begin{aligned} \Delta A_x &= -\mu j_x \\ \Delta A_y &= -\mu j_y \\ \Delta A_z &= -\mu j_z \end{aligned}$$

allgemeine Definition: $\Delta\vec{u} = \nabla(\operatorname{div}\vec{u}) - \operatorname{rot}(\operatorname{rot}\vec{u})$

z.B. $\vec{u} = \begin{pmatrix} 2x^2+4y^3+z^2 \\ -z^4 \\ 4y^5-z^3 \end{pmatrix}$

6. Aufgabe

a) $$\left(\Delta - \cancel{\mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}}\right) \vec{A} = \cancel{\mu_0 \varepsilon_0 \nabla \frac{\partial}{\partial t} \Phi} - \mu_0 \vec{j}$$

$$\Delta \vec{A} = -\mu_0 \vec{j}$$

Spezialfall kartesische Koordinaten:

$$\left.\begin{aligned} \Delta A_x &= -\mu_0 j_x \\ \Delta A_y &= -\mu_0 j_y \end{aligned}\right\} \quad j_x = j_y \equiv 0 \Rightarrow A_x = A_y \equiv 0$$

$$\Delta A_z = -\mu_0 j_z$$

$$A_z = \frac{\mu}{4\pi} \iiint_V \frac{j_z}{|\vec{r} - \vec{r}\,'|} \, d^3 r'$$

b) ges: Lösung der DGL

Lös: $$|\vec{r} - \vec{r}\,'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$A_z = \frac{\mu}{4\pi} \int_{-l}^{l} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{j_z \, \delta(x') \, \delta(y')}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')}} \, dx' \, dy' \, dz'$$

Exkurs:

$$\int_{-\infty}^{\infty} f(x) \, \delta(x - x') \, dx' = f(x')$$

$$\int_{-\infty}^{\infty} \delta(x) \, dx = 1$$

$$= \frac{\mu}{4\pi} I \int_{-l}^{l} \frac{1}{\sqrt{\underbrace{(x\quad)^2 + (y\quad)^2}_{R^2} + (\underbrace{z-z'}_{u})^2}}\, dz'$$

Subst: $u = z - z'$

$$\frac{du}{dz'} = -1 \Leftrightarrow dz' = -du$$

$$\Rightarrow A_z = -\frac{\mu_0}{4\pi} I \int_{z+l}^{z-l} \frac{1}{\sqrt{R^2+u^2}}\, du =$$

$$= -\frac{\mu_0}{4\pi} I \ln\left(u' + \sqrt{R^2+u'^2}\right)\Big|_{z+l}^{z-l} =$$

$$= -\frac{\mu_0}{4\pi} I \ln\left(\frac{z-l+\sqrt{R^2+(z-l)^2}}{z+l+\sqrt{R^2+(z+l)^2}}\right)$$

c) $$\lim_{l\to\infty} A_z = \lim_{l\to\infty} -\frac{\mu_0}{4\pi} I \ln\left(\frac{z/l - 1 + \sqrt{\left(\frac{R}{l}\right)^2 + \left(\frac{z}{l}-1\right)^2}}{z/l + 1 + \sqrt{\left(\frac{R}{l}\right)^2 + \left(\frac{z}{l}+1\right)^2}}\right)$$

$$= -\frac{\mu_0}{4\pi} \ln\left(\frac{0}{2}\right) = \rightarrow \infty$$

$\Rightarrow$ divergiert

7. Aufgabe

a) $\vec{A} = \begin{pmatrix} 0 \\ 0 \\ A_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x^2 + y^2 \end{pmatrix}$

ges: $\vec{B}$

Lös: $\vec{B} = \text{rot}\,\vec{A} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ x^2 + y^2 \end{pmatrix} = \begin{pmatrix} 2y \\ -2x \\ 0 \end{pmatrix}$

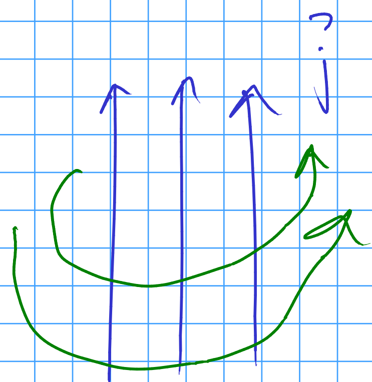
b) $\vec{A}$ sollte zur Berechnung verwendet werden

c) $\vec{B}$ homogen $\Rightarrow$ $\vec{B} \neq \vec{B}(x,y,z)$

$\text{rot}\,\vec{H} = \text{rot}\left(\frac{\vec{B}}{\mu}\right) = 0$

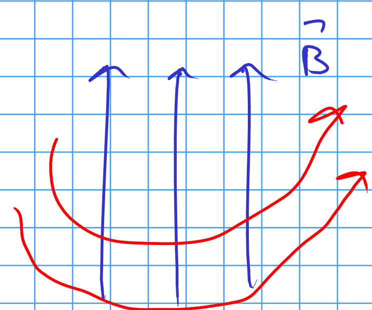
$\text{rot}\,\vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \Rightarrow \vec{j} = 0$

d)



$\Rightarrow$ rechte Handregel

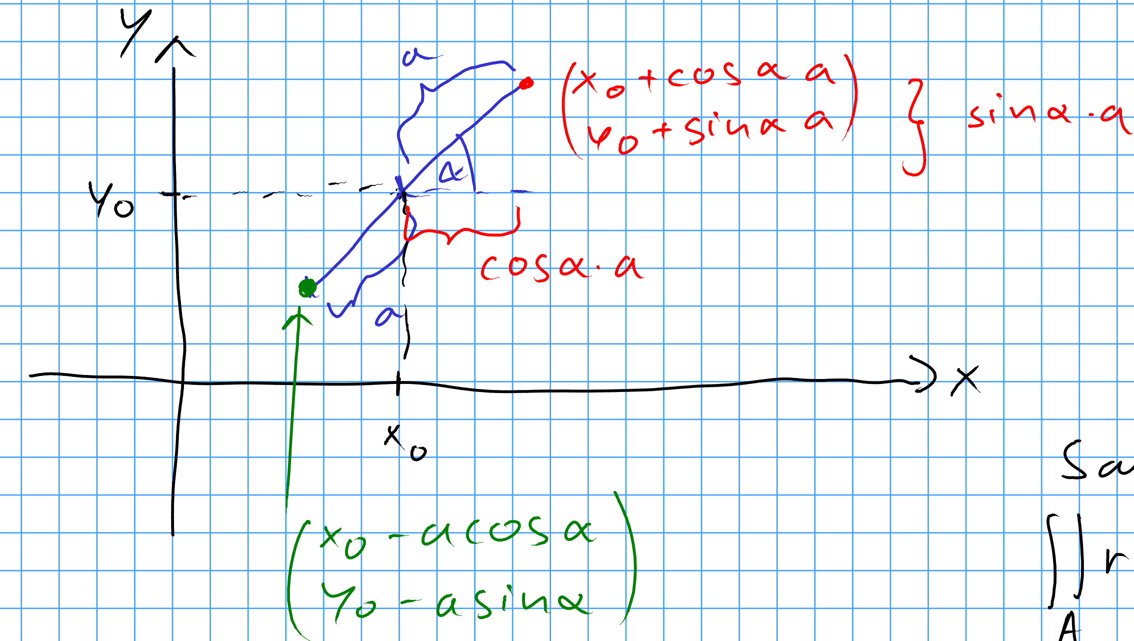
e)



$\text{rot}\,\vec{A} = \vec{B}$

$\text{rot}\,\vec{H} = \vec{j}$

f)



Satz v. Stokes:

$$\iint_A \operatorname{rot} \vec{u}\, d\vec{a} = \int_{\partial A} \vec{u}\, d\vec{r}$$

$$U_{ind} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \iint_A \vec{B}\, d\vec{a} = -\frac{d}{dt} \iint_A \operatorname{rot} \vec{A}\, d\vec{a} =$$

$$= -\frac{d}{dt} \int_{\partial A} \vec{A}\, d\vec{r}$$

g) $\partial A = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$

$$\int_{\partial A} \vec{A}\, d\vec{r} = \underbrace{\overbrace{\int_{\gamma_1} \vec{A}\, d\vec{r}}^{\equiv 0}} + \int_{\gamma_2} \vec{A}\, d\vec{r} + \overbrace{\int_{\gamma_3} \vec{A}\, d\vec{r}}^{\equiv 0} + \int_{\gamma_4} \vec{A}\, d\vec{r}$$

$$\left[\int_{\gamma_1} \vec{A}\, d\vec{r} = \int_0^{t_0} \vec{A}(\gamma_1(t))\, \dot{\gamma}_1(t)\, dt \right]$$ EM!

$$\gamma_1 = \begin{pmatrix} x_0 - a\cos(\omega t) \\ y_0 - a\sin(\omega t) \\ 0 \end{pmatrix} + s \begin{pmatrix} 2a\cos(\omega t) \\ 2a\sin(\omega t) \\ 0 \end{pmatrix}$$

$s \in [0;1]$

$$\gamma_2 = \begin{pmatrix} x_0 + a\cos(\omega t) \\ y_0 + \sin(\omega t) \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ l \end{pmatrix}$$

$s \in [0,1]$

$$\gamma_3 = \begin{pmatrix} x_0 + a\cos(\omega t) \\ y_0 + a\sin(\omega t) \\ l \end{pmatrix} + s \begin{pmatrix} -2a\cos(\omega t) \\ -2a\sin(\omega t) \\ 0 \end{pmatrix}$$

$s \in [0,1]$

$$\gamma_4 = \begin{pmatrix} x_0 - a\cos(\omega t) \\ y_0 - a\sin(\omega t) \\ l \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ -l \end{pmatrix}$$

$s \in [0,1]$

Begrü: $\vec{A}(\gamma_1(t))^T \dot{\gamma}_1 = \begin{pmatrix} 0 \\ 0 \\ (x_0 - a\cos(\omega t))^2 + (y_0 - a\sin(\omega t))^2 \end{pmatrix}^T \begin{pmatrix} \sim \\ \sim \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\int_{\gamma_2} \vec{A}(\gamma_2(t))\, \dot{\gamma}_2(t)\, dt = l\left[(x_0 + a\cos(\omega t))^2 + (y_0 + a\sin(\omega t))^2\right]$$

$$\int_{\gamma_4} \vec{A}(\gamma_4(t))\, \dot{\gamma}_4(t)\, dt = -l\left[(x_0 - a\cos(\omega t))^2 + (y_0 - a\sin(\omega t))^2\right]$$

Es gilt: $\partial A = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$

$$\begin{aligned}
U_{ind} &= -\frac{d}{dt}\int_{\partial A} \vec{A}\mathrm{d}\vec{r} = -\frac{d}{dt}\left[\int_{\gamma_2} \vec{A}\mathrm{d}\vec{r} + \int_{\gamma_4} \vec{A}\mathrm{d}\vec{r}\right] \\
&= -l\frac{d}{dt}\left\{\left[(x_0 + a\cos(\omega t))^2 + (y_0 + a\sin(\omega t))^2\right] - \left[(x_0 - a\cos(\omega t))^2 + (y_0 - a\sin(\omega t))^2\right]\right\} = \\
&= -l\frac{d}{dt}\left[4ax_0\cos(\omega t) + 4ay_0\sin(\omega t)\right] = \\
&= 4a\omega l\left[y_0\cos(\omega t) - x_0\sin(\omega t)\right]
\end{aligned}$$