EMF Tutorübung, Blatt 6 – 17.12.2010

Wiederholung: Lösungsverfahren von Potential-RWP

→ Green-Funktion: $\Delta G(\vec{r}, \vec{r}\,') = -\frac{1}{\varepsilon}\delta(\vec{r}-\vec{r}\,')$ $\forall \vec{r} \in V$

$G(\vec{r}, \vec{r}\,') = 0 \quad \forall \vec{r} \notin V$

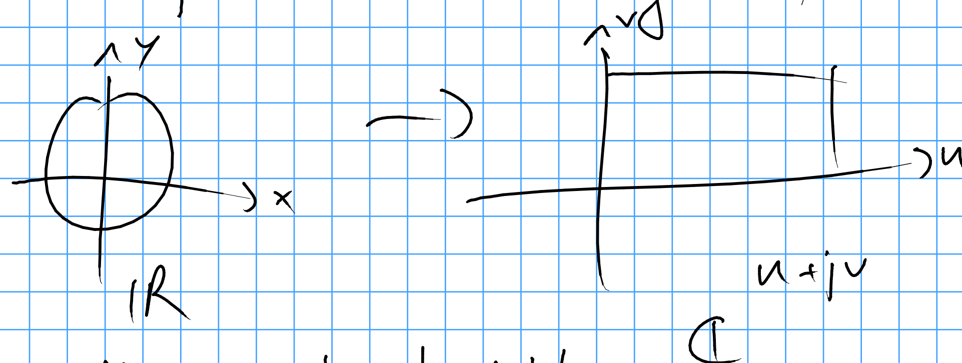
→ muss nur einmal für geg. Geometrie berechnet werden

$$\phi(\vec{r}) = \iiint_V G(\vec{r}, \vec{r}\,')\, \rho(\vec{r}\,')\, d^3r'$$

→ Möglichkeiten zur Bestimmung:

⇒ Spiegelladungsmethode

⇒ konforme Abbildung (2D)



⇒ Orthogonalentwicklung

Orthogonalentwicklung: $\underset{\sim}{A} v = \lambda v$

$\uparrow$ Eigenvektor

$$\Delta \Phi = -\rho / \varepsilon$$

$$-\Delta b = \lambda b$$

Seperationsansatz: $b(x_1, x_2, x_3) = b_1(x_1)\, b_2(x_2)\, b_3(x_3)$

$$-\left(b_1'' b_2 b_3 + b_1 b_2'' b_3 + b_1 b_2 b_3''\right) = \lambda b_1 b_2 b_3 \quad | : b$$

$$-\left(\underbrace{\frac{b_1''(x_1)}{b_1(x_1)}}_{\text{const.}} + \underbrace{\frac{b_2''(x_2)}{b_2(x_2)}}_{\text{const.}} + \underbrace{\frac{b_3''(x_3)}{b_3(x_3)}}_{\text{const.}}\right) = \lambda$$

(I) $-\dfrac{b_1''(x_1)}{b_1(x_1)} = \lambda_1$ (II) $-\dfrac{b_2''(x_2)}{b(x_2)} = \lambda_2$ (III) $-\dfrac{b_3''(x_3)}{b_3(x_3)} = \lambda_3$

$$\lambda_1 + \lambda_2 + \lambda_3 = \lambda$$

$b_i(x_i) = A_i \sin(k_i x_i)$ $\quad \sin(k_i L) = 0$

$\Leftrightarrow k_i L = n\pi$

$\Rightarrow$ RB berücksichtigen (Finden der Konstanten)

$\Rightarrow$ Normierung $\quad \langle x(t), y(t)\rangle = \int_{-\infty}^{\infty} x(t)\, y(t)^* \, dt$

$$\|x\|^2 = \langle x, x\rangle$$

$$\int_{-\infty}^{\infty} b_1(x_i)\, b_i(x_i)^* \, dt \overset{!}{=} 1$$

Normierungsbedingung

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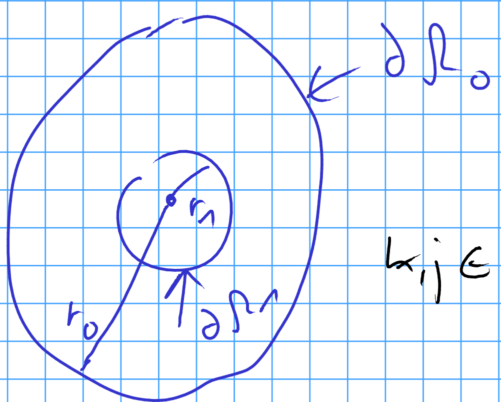

$k, j \in \{0; 1\}$

a) Laplace-Gl. $\Delta \phi_k(\vec{r}) = 0$

b) $\phi_{ges}(\vec{r}) = V_0 \phi_0(\vec{r}) + V_1 \phi_1(\vec{r})$

$$\phi_k(\vec{r})\Big|_{\partial\Omega_k} = 1$$

$$\phi_k(\vec{r})\Big|_{\partial\Omega_j} = 0$$

$$\phi_k(\vec{r})\Big|_{\partial\Omega_j} = \underbrace{\delta_{jk}}_{\text{Kronecker}}$$

c) radiale Abhängigkeit

d) Erinnerung: $\Delta f = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \dots$

$\Delta \phi = 0$

(I) $\phi_k(r) = C_0$

(II) $\phi_k(r) = C_1 \cdot r + C_0$

(III) $\phi_k(r) = C_1 \cdot \frac{1}{r} + C_0$

(I) $\left.\begin{aligned}\phi_0(r=r_0) &= 1 \\ \phi_0(r=r_1) &= 0\end{aligned}\right\} \Rightarrow$ (I) kann nicht gesuchte Lösung sein

(II) $\Delta\phi_K = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 c_1\right) = \frac{2}{r} c_1 \neq 0 \quad \forall r$

(III) $\phi_K(r) = c_1 \cdot \frac{1}{r} + c_0$

$$\Delta\phi_K = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \left(-\frac{c_1}{r^2}\right)\right) = 0$$

e) (I) $\phi_0(r_0) = 1$ (II) $\phi_1(r_1) = 1$

$\phi_0(r_1) = 0$ $\phi_1(r_0) = 0$

(I) $c_1 \cdot \frac{1}{r_0} + c_0 = 1$

$$c_1 \cdot \frac{1}{r_1} + c_0 = 0 \Rightarrow c_0 = -\frac{c_1}{r_1}$$

$$c_1 \cdot \frac{1}{r_0} - \frac{c_1}{r_1} = 1 \Leftrightarrow c_1\left(\frac{1}{r_0} - \frac{1}{r_1}\right) = 1$$

$$c_1 = \frac{r_0 r_1}{r_1 - r_0}$$

$$c_0 = -\frac{r_0}{r_1 - r_0}$$

(II) $C_1 \cdot \frac{1}{r_1} + C_0 = 1$

$$C_1 \cdot \frac{1}{r_0} + C_0 = 0 \quad \Rightarrow \quad C_0 = -\frac{C_1}{r_0}$$

$$C_1 \cdot \frac{1}{r_1} - \frac{C_1}{r_0} = 1 \iff C_1\left(\frac{1}{r_1} - \frac{1}{r_0}\right) = 1$$

$$\Rightarrow \quad C_1 = \frac{r_1 r_0}{r_0 - r_1}$$

$$C_0 = -\frac{r_1}{r_0 - r_1}$$

$$\phi_{ges} = \left(\frac{r_0 r_1}{r_1 - r_0} \cdot \frac{1}{r} - \frac{r_0}{r_1 - r_0}\right) \cdot V_0 +$$
$$\left(\frac{r_1 r_0}{r_0 - r_1} \cdot \frac{1}{r} - \frac{r_1}{r_0 - r_1}\right) \cdot V_1$$

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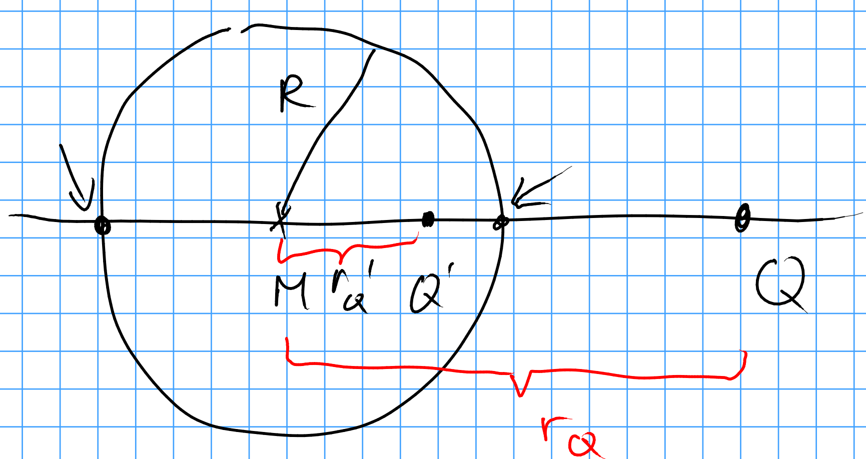
$$Q' = -kQ$$



a) $\vec{G}(\vec{r}, \vec{r}\,')$

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon}\left[\frac{Q}{|\vec{r} - \vec{r}_Q|} - \frac{kQ}{|\vec{r} - \vec{r}_Q\,'|}\right]$$

$$|\vec{r} - \vec{r}_Q|^2 = |\vec{r}|^2 + |\vec{r}_Q|^2 - 2|\vec{r}||\vec{r}_Q|\cos(\gamma)$$

$$\phi(\vec{r} = R\vec{e}_r) = 0$$

$$(\mathrm{I}) \quad \cancel{\frac{1}{4\pi\varepsilon}} \left[\frac{Q}{r_Q - R} - \frac{kQ}{R - r_Q'} \right] = 0$$

$$(\mathrm{II}) \quad \cancel{\frac{1}{4\pi\varepsilon}} \left[\frac{Q}{r_Q + R} - \frac{kQ}{R + r_Q'} \right] = 0$$

$$(\mathrm{I}') \quad \frac{1}{r_Q - R} - \frac{k}{R - r_Q'} = 0 \Rightarrow k = \frac{R - r_Q'}{r_Q - R}$$

$$(\mathrm{II}') \quad \frac{1}{r_Q + R} - \frac{R - r_Q'}{(r_Q - R)(R + r_Q')} = 0$$

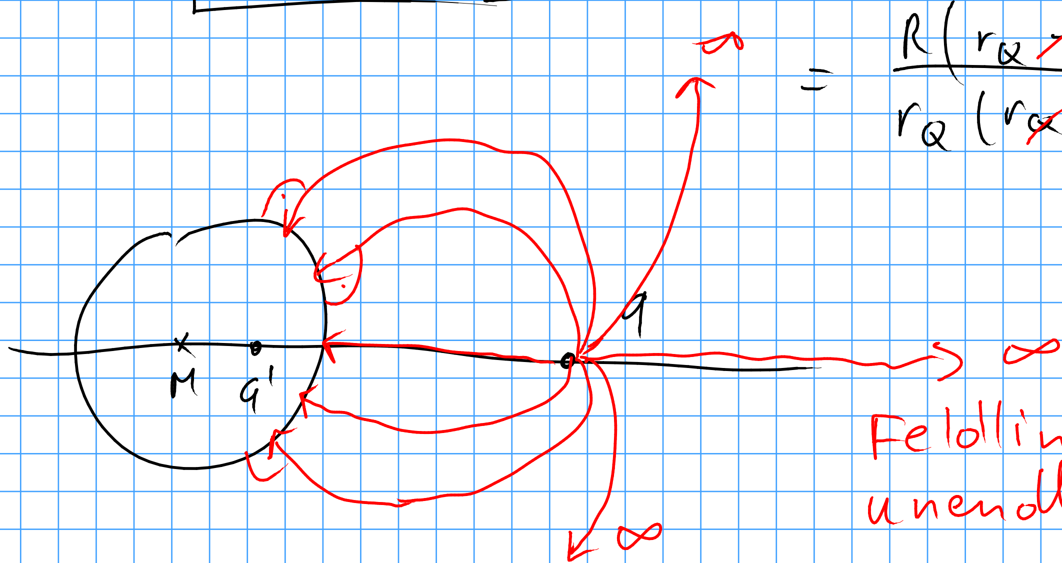
$$(r_Q - R)(R + r_Q') - (R - r_Q')(r_Q + R) = 0$$

$$R r_Q + r_Q r_Q' - R^2 - R r_Q' - (r_Q R + R^2 - r_Q' r_Q - R r_Q') = 0$$

$$2 r_Q r_Q' - 2R^2 = 0$$

$$\boxed{r_Q' = \frac{R^2}{r_Q}} \qquad k = \frac{R - \frac{R^2}{r_Q}}{r_Q - R} = \frac{R r_Q - R^2}{r_Q^2 - R r_Q} = \frac{R(\cancel{r_Q - R})}{r_Q(\cancel{r_Q - R})} = \frac{R}{r_Q}$$

b)



Feldlinien auch ins unendliche, da $|q| > |q'|$

c) sieht analog aus (Hintergrund: Symmetrie $G(\vec{r}, \vec{r}') = G(\vec{r}', \vec{r})$)

d) $Q' = \underbrace{-\left(\frac{R}{r_a}\right)}_{k} \underbrace{\iiint}_{V} \rho(\vec{r})\, d^3 r$