ST2 – Tutorübung, Blatt 5 – 15.6.2011

Jordan-Normalform: behandelt den Fall, dass zwei gleiche EW vorliegen

bisher: $\underset{\sim}{A} = \underset{\sim}{Q}\,\Lambda\,\underset{\sim}{Q}^{-1}$ $\qquad \underset{\sim}{Q} = (q_1 \;\; q_2)$

$\lambda_1 = \lambda_2 \iff q_1 = q_2$

$\iff Q^{-1}$ nicht existent

$\underset{\sim}{A} = \underset{\sim}{Q}'\,\underset{\sim}{\Lambda}'\,\underset{\sim}{Q}^{-1}$ $\qquad$ hier $A = \mathbb{R}^{2\times 2}$

$$\underset{\sim}{\Lambda}' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\text{nilpotent}}$$

$[B^n = 0]$

$$e^{t\underset{\sim}{\Lambda}'} = \sum_{k=0}^{\infty} \frac{t^k \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^k}{k!} =$$

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^k = \begin{pmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{pmatrix}$$

Hinweis: math. Induktion

$$= \sum_{k=0}^{\infty} \frac{t^k \begin{pmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{pmatrix}}{k!} =$$

$$= \sum_{k=0}^{\infty} \frac{\begin{pmatrix} (t\lambda)^k & t^k k \lambda^{k-1} \\ 0 & (t\lambda)^k \end{pmatrix}}{k!} =$$

$$= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{(t\lambda)^k}{k!} & \sum_{k=0}^{\infty} \frac{t^k k \lambda^{k-1}}{k!} \\ 0 & \sum_{k=0}^{\infty} \frac{(t\lambda)^k}{k!} \end{pmatrix} =$$

$$= \begin{pmatrix} e^{t\lambda} & t \sum_{k=0}^{\infty} \frac{(t\lambda)^{k-1}}{(k-1)!} \\ 0 & e^{t\lambda} \end{pmatrix} = \begin{pmatrix} e^{t\lambda} & t e^{t\lambda} \\ 0 & e^{t\lambda} \end{pmatrix}$$

$\Rightarrow$ allg. Lösung:

$$x(t) = c_1 q_1 e^{\lambda t} \boxed{+ c_2 t q_1 e^{\lambda t}} + c_2 q_2 e^{\lambda t}$$

q_1: EV zum EW λ

$$q_2 = \begin{pmatrix} -a_{12} \\ \frac{a_{11} - a_{12}}{2} - 1 \end{pmatrix}$$

Fall: $\lambda_1, \lambda_2 \in \mathbb{C}$ $\qquad ax^2 + bx + c = 0$

$\lambda_1 = \lambda_2^* = \lambda$ (Fundamentalsatz d. Algebra)

$q_1 = q_2^* = q$ $\qquad c = c_1 = c_2^*$

$\lambda = \alpha + j\beta$ $\qquad x = u + jv$

$y = u - jv$

$$x(t) = c_1 q_1 e^{\lambda_1 t} + c_2 q_2 e^{\lambda_2 t} =$$

$$= c_1 q e^{\lambda t} + c_2 q^* e^{\lambda^* t} =$$

$$= c q e^{\lambda t} + c^* q^* e^{\lambda^* t} =$$

$$= 2\,\mathrm{Re}\left(c q e^{\lambda t}\right) =$$

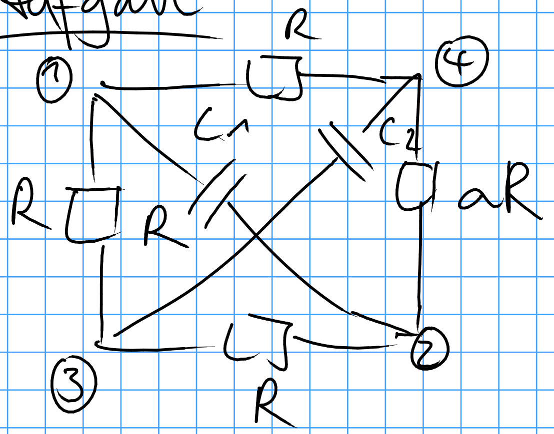
$$e^{\lambda t} = e^{(\alpha + j\beta)t} = e^{\alpha t} \cdot e^{j\beta t} =$$

$$= e^{\alpha t}\left(\cos(\beta t) + j\sin(\beta t)\right)$$

$c = c_1 + jc_2 \qquad q = q_r + jq_i$

$$x(t) = 2e^{\alpha t}\Big[\cos(\beta t)\left[c_1 q_r - c_2 q_i\right] - \sin(\beta t)\left[c_1 q_i - c_2 q_r\right]\Big]$$

Aufgabe



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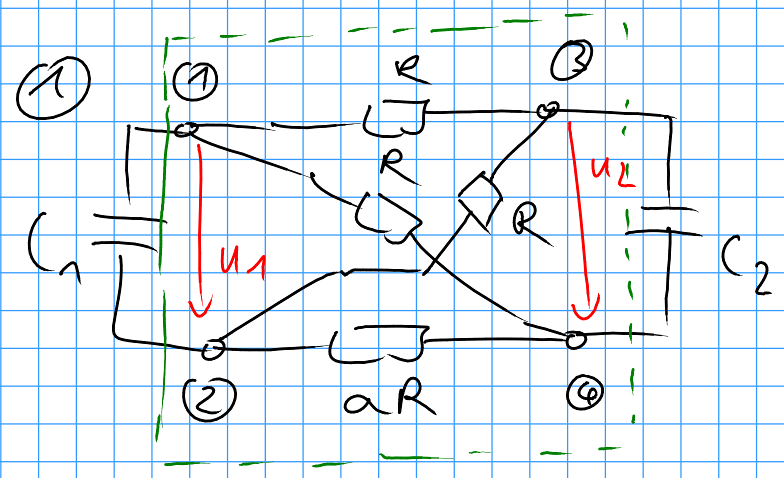
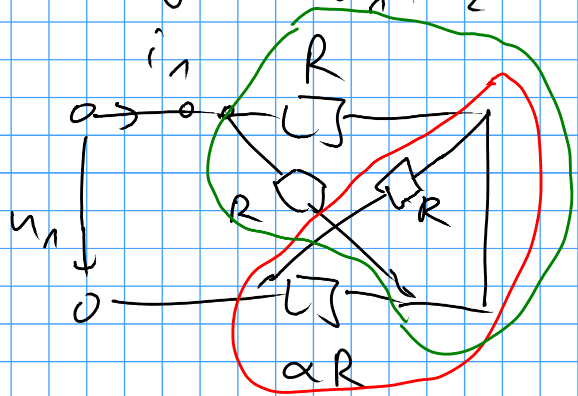


2. $\underline{x} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

3. $\underline{i} = \underset{\sim}{G}\,\underline{u} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

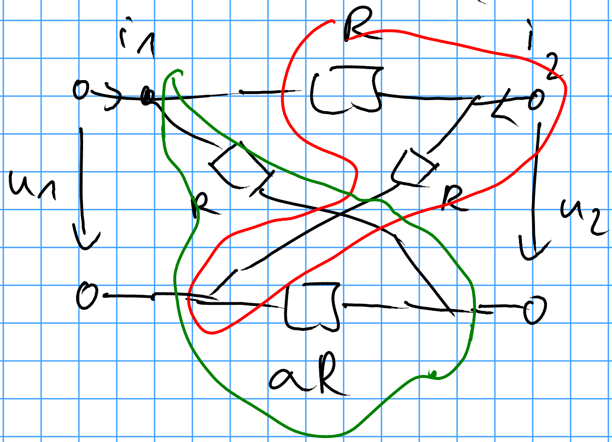
$$\Rightarrow \quad g_{11} = \frac{i_1}{u_1}\Big|_{u_2=0} = \frac{1}{R\|R + R\|\alpha R} =$$



$$= \frac{1}{\frac{R}{2} + \frac{\alpha R^2}{(\alpha+1)R}} =$$

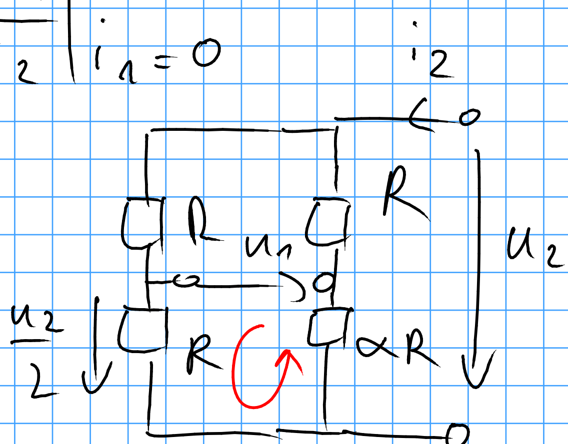
$$= \frac{2(1+\alpha)}{R(3\alpha+1)}$$

$$\underline{u} = \underset{\sim}{R}\,\underline{i} = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$$



$$r_{11} = \left.\frac{u_1}{i_1}\right|_{i_2=0} = 2R \,||\, (\alpha+1)R = \frac{2R\,(1+\alpha)R}{R(\alpha+1)+2R} = \frac{2(1+\alpha)R}{(\alpha+3)}$$

$$r_{12} = \left.\frac{u_1}{i_2}\right|_{i_1=0}$$



$$u_1 = \frac{u_2}{2} - u_2 \cdot \frac{\alpha R}{R+\alpha R}$$

$$i_2 = \frac{u_2}{2R \,||\, (R+\alpha R)}$$

$$\Rightarrow r_{12} = \frac{R(1+\alpha)}{3+\alpha}$$

$r_{21} = r_{12}$ (Hintergrund: Reziprozität einer passiven Schaltung: $\underset{\sim}{R} = \underset{\sim}{R}^T$)

$$r_{22} = \frac{u_2}{i_2} = r_{11} = \frac{2R(1+\alpha)}{3+\alpha} \qquad \left(\text{Symmetrie: } \underset{\sim}{R} = \underset{\sim}{P}\, R\, \underset{\sim}{P}\right) \leftarrow \text{ST1}$$

$$\underset{\sim}{G} = \underset{\sim}{R}^{-1} = \frac{1}{\det \underset{\sim}{R}} \begin{pmatrix} r_{22} & -r_{12} \\ -r_{21} & r_{11} \end{pmatrix} =$$

$$= \frac{1}{R(3\alpha+1)} \begin{pmatrix} 2(1+\alpha) & -(1-\alpha) \\ -(1-\alpha) & 2(1+\alpha) \end{pmatrix}$$

(5) $G = \operatorname{diag}(\lambda_1, \lambda_2)$

$$\boxed{\alpha = 1}$$

(6) $$\underset{\sim}{A} = \begin{pmatrix} \beta & 2\beta - 10 \\ \frac{1}{2}\beta & 2\beta \end{pmatrix}$$

$$\det(\lambda E_2 - A) \overset{!}{=} 0$$

$$\begin{vmatrix} \lambda - \beta & 10 - 2\beta \\ -\frac{1}{2}\beta & \lambda - 2\beta \end{vmatrix} \overset{!}{=} 0$$

$$(\lambda - \beta)(\lambda - 2\beta) + \frac{1}{2}\beta(10 - 2\beta) \overset{!}{=} 0$$

$$\lambda^2 - 3\beta\lambda + 5\beta + \beta^2 \overset{!}{=} 0$$

$$\lambda_{1/2} = \frac{3\beta \pm \sqrt{9\beta^2 - 20\beta - 4\beta^2}}{2} =$$

$$= \frac{3}{2}\beta \pm \frac{1}{2}\sqrt{5\beta^2 - 20\beta}$$

⑦ $D = 0 \iff 5\beta^2 - 20\beta = 0$

$\beta = 0 \vee \beta = 4$

⑧ Jordan - Normaltransformation

$$\underset{\sim}{A} = \Big|_{\beta=4} \begin{pmatrix} 4 & -2 \\ 2 & 8 \end{pmatrix} \qquad \lambda_{1/2} = 6$$

Bestimmung des EV zum geg. EW:

$$(A - \lambda E_2)\, q = 0$$

$$\begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix} q = 0 \qquad q_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$q_2 = \begin{pmatrix} -a_{12} \\ \frac{a_{11} - a_{22}}{2} - 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$Q' = \begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix}$$

allg. $\underline{x}(t) = c_1 e^{6t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 t e^{6t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$+ c_2 e^{6t} \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

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$$D < 0 \iff 5\beta^2 - 20\beta < 0$$

$$\beta(5\beta - 20) < 0$$

1. Fall: $\beta < 0 \wedge \beta > 4$ $\lightning$

2. Fall: $\beta > 0 \wedge \beta < 4$

$$\Rightarrow \beta \in (0; 4)$$

$\Rightarrow$ harmonischer Oszillator existiert also, falls

$\triangleright$ $\mathrm{Re}(\lambda_{1/2}) > 0$

$\triangleright$ $\mathrm{Im}(\lambda_{1/2}) \neq 0$

(10) $\beta = 2$

$$\underset{\sim}{A} = \begin{pmatrix} 2 & -6 \\ 1 & 4 \end{pmatrix}$$

$$\leadsto \lambda_{1,2} = 6 \pm j\sqrt{5}$$

$\hookrightarrow$ instabiler Strudel